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Abstract. We study the dependence on time delay τ of the average echo power $I(\tau)$ resulting from the reflection of a quasi-monochromatic outgoing pulse of radiation by a multiscale random surface, modelled as a fractal with Hausdorff–Besicovitch dimension $D+1$ between 2 and 3. In the 'weak fractal' limit where the fractal height variations are small, the reflection is paraxial and $I(\tau)$ decays as $\tau^{-(3-D)}$, for both statistically isotropic and 'corrugated' surfaces. This behaviour contrasts with a smooth random surface, for which $I(\tau)$ decays exponentially.

1. Introduction

Our purpose here is to study the reflection of an outgoing quasi-monochromatic pulse of radiation (acoustic or electromagnetic) by a random surface whose roughness extends down through scales much smaller than the wavelength λ. It is natural to model such surkces by fractals (Mandelbrot 1977), that is as geometric objects with a hierarchy of structure extending infinitely small, whose Hausdorff–Besicovitch dimension lies between 2 and 3; this means the roughness is so extreme that a finite region of surface has infinite area but zero volume. We shall write the dimension as $D+1$, so that D is the dimension of the fractal function obtained by cutting the surface along a line (i.e. $1<D<2$). (Examples of such functions were studied in detail by Berry and Lewis (1980).) A smooth surface has dimension 2 and $D=1$.

In a previous paper, Berry (1979) introduced the study of waves encountering fractals ('diffractals') as a new regime in wave theory, distinguished by the fact that the usual short-wave limit (e.g. geometrical optics) is never attained, because the result of taking $\lambda \to 0$ is to expose ever-finer detail rather than something ultimately smooth. Instead, the short-wave limit for diffractals is characterised by D-dependent wave properties (e.g. scaling laws). The system studied by Berry (1979) was the diffraction of a plane monochromatic wave by a random fractal phase screen, and it was found that the simplest diffractal phenomenfa occurred in the behaviour of the intensity fluctuations far from the screen.

What we show here is that by taking the incident radiation as an outgoing quasi-monochromatic pulse rather than a plane monochromatic wave, diffractal phenomena occur in the average intensity itself, whose calculation is considerably simpler than that of the statistics of fluctuations in intensity. In particular, in a certain

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parameter regime the echo tail decays in time according to a D-dependent power law when the rough surface is a fractal, in contrast with the exponential decay for smooth random surfaces.

2. Surface statistics

We represent the surface by an ensemble of Gaussian random fractal 'height' functions $h(\boldsymbol{R})$ giving its deviation from the plane $\boldsymbol{R}=(X, Y)$. Denoting ensemble averages by angle brackets, we specify the statistics of h by the *mean height* $\langle h\rangle$, which we take to be zero, the *mean square height*,

$$\langle h^2\rangle \equiv H^2 \tag{1}$$

and its *mean square increment*

$$\langle (h(\boldsymbol{R}_0+\boldsymbol{R})-h(\boldsymbol{R}_0))^2\rangle \equiv \Delta(\boldsymbol{R}). \tag{2}$$

We shall consider *isotropic randomness*, in which Δ depends only on $R\equiv(X^2+Y^2)^{1/2}$, and *corrugated randomness*, in which Δ depends only on X.

The fractality of h is embodied in the behaviour of $\Delta(\boldsymbol{R})$ as $\boldsymbol{R}\to 0$. For a $(D+1)$-dimensional surface, we must have

$$\Delta(\boldsymbol{R}) \xrightarrow{(\boldsymbol{R}\to 0)} \begin{cases} L^{2D-2}|X|^{4-2D} & \text{corrugated} \\ L^{2D-2}R^{4-2D} & \text{isotropic} \end{cases} \quad 1<D<2. \tag{3}$$

Here L is the 'topothesy', a length characterising the strength of the fractal roughness and equal to the distance (R or X) over which chords joining points of the surface have an RMS slope of one radian. A proof that Gaussian random functions with the incremental behaviour (3) do in fact represent surfaces whose cuts have dimension D is given in appendix 1 of Berry (1979), following Orey (1970).

The large-scale 'geography' of h is embodied in the behaviour of $\Delta(\boldsymbol{R})$ as $R\to\infty$, namely

$$\Delta(\boldsymbol{R})\to 2H^2 \qquad \text{as } R\to\infty \text{ or } |X|\to\infty. \tag{4}$$

A simple model incorporating these limiting forms (written just for the isotropic case) is

$$\Delta(R)=2H^2[1-\exp(-L^{2D-2}R^{4-2D}/2H^2)]. \tag{5}$$

For this model, the *correlation length* l, describing the outer scale of the roughness, is

$$l=L(H/L)^{1/(2-D)}. \tag{6}$$

Here we shall consider only surfaces for which $H\gg L$, so that $l\gg L$ and the scales of fractality and geography can be clearly separated.

An important special case is $D=1.5$, called the 'Brownian fractal' because (3) implies that Δ varies linearly for small R or X like the increments of a coordinate h of a particle undergoing Brownian motion for time R or X. In the limiting case $D\to 1$, corresponding to a smooth surface, the topothesy L disappears from (3), and we can write instead (for the isotropic case)

$$\Delta(R)\xrightarrow{R\to 0}\beta^2R^2 \tag{7}$$

where β is the RMS surface slope. (5) and (6) then become

$$\Delta(R) = 2H^2[1-\exp(-\beta^2 R^2/2H^2)] \tag{8}$$

and

$$l = H/\beta; \tag{9}$$

we shall assume $\beta \gg 1$, so that $l \gg H$.

The Gaussian randomness of h implies the following exponential averaging rule, which we shall use in the diffraction theory of the next section; for any constant b,

$$\langle \exp[\mathrm{i}b(h(\boldsymbol{R}_1) - h(\boldsymbol{R}_2))]\rangle = \exp(-\tfrac{1}{2}b^2\Delta(\boldsymbol{R}_1 - \boldsymbol{R}_2)). \tag{10}$$

3. Diffraction integrals

Figure 1 illustrates the geometry of the diffraction problem we shall study. A source–receiver S, situated at height Z above the $\boldsymbol{R}$ plane defining the mean surface, emits a spherical pulse $\phi(r, t)$ whose dependence on time t and distance r from S is

$$\phi(r, t) = F(t - r/c)/r \tag{11}$$

where c is the wave speed. For F we take the quasi-monochromatic form

$$F(t) = \mathrm{e}^{\mathrm{i}\omega t}a(t) \tag{12}$$

where ω is the carrier frequency and $a(t)$ the slowly varying pulse envelope. The wave number k and wavelength λ are defined by

$$\omega \equiv ck \equiv 2\pi c/\lambda. \tag{13}$$

For detailed calculations we shall use the Gaussian envelope function

$$a(t) = a_0 \exp(-c^2t^2/\sigma^2) \tag{14}$$

where σ is twice the RMS spatial pulse length of the power envelope a^2. Quasi-monochromaticity will be ensured by taking $\lambda \ll \sigma$.

This pulse is diffracted by the surface and the echo wavefunction is ψ received back at S. We are interested in the statistics of ψ over the ensemble of random surfaces $h(\boldsymbol{R})$,

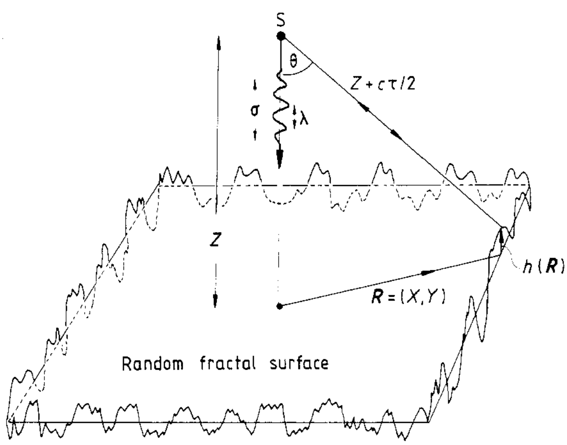

Figure 1. Geometry of diffractal echo problem.

in particular the intensity

$$I(\tau) \equiv \langle|\psi(\tau)|^2\rangle \tag{15}$$

as a function of time delay τ measured from the time $t = 2Z/c$ at which the mid-echo would be received after reflection from a plane mirror.

Exact determination of the echo $\psi(\tau)$ is an unsolved problem for any non-trivial surface $h(\boldsymbol{R})$. We shall use the Kirchhoff approximation, in which ψ is expressed as a superposition of wavelets from secondary sources $\boldsymbol{R}$ on the surface. Now the Kirchhoff approximation neglects multiple scattering and shadowing of one part of the surface by another (as seen by S), and yet we are considering fractal surfaces consisting of jagged cliffs with infinite RMS slope, for which multiple scattering and shadowing will surely occur unless the main incident and scattered contributions make small angles with the normal to the mean surface plane. Therefore we must make the additional 'Fresnel' or 'paraxial' approximation to the diffraction integral, namely (see e.g. Berry 1972, 1973)

$$\psi(t) = \frac{-1}{2cZ^2} \iint \mathrm{d}\boldsymbol{R}\, F'\left(\tau + \frac{2h(\boldsymbol{R})}{c} - \frac{R^2}{cZ}\right) \tag{16}$$

where the prime denotes differentiation of F with respect to its argument. This formula will be valid for time delays τ for which the scattering angle θ, given by (cf figure 1)

$$\cos\theta = Z/(Z + \tfrac{1}{2}c\tau), \tag{17}$$

is small enough. Because (16) involves expanding up to order θ^2, the precise condition for paraxiality is

$$\frac{\theta^4}{4!} \approx \frac{1}{24}\left(\frac{c\tau}{2Z}\right)^2 \ll 1. \tag{18}$$

A rigorous analysis of this intuitive 'paraxial' justification of the Kirchhoff approximation appears to be extremely difficult. Evidence that the Kirchhoff procedure is at least not internally inconsistent when applied to a fractal surface is given by Berry (1979) who shows that if the topothesy is small enough (a condition later shown to be implied by (18)) then most radiation is scattered paraxially.

A pleasant feature of the approximation (16) is that when applied to a flat mirror ($h = 0$) it gives the exact wave

$$\chi(\tau) = -F(\tau)/2Z \tag{19}$$

corresponding to the reflection with sign reversal of the wave (11) from an image $2Z$ below S.

In using (16) to calculate the echo intensity $I(\tau)$ from (15), a useful simplification is to note that, because of quasi-monochromaticity, the derivative of the pulse function (12) can be written as

$$F'(t) \approx \mathrm{i}\omega\, \mathrm{e}^{\mathrm{i}\omega t} a(t). \tag{20}$$

The intensity is therefore given by

$$I(\tau) = \left(\frac{k}{2\pi Z^2}\right)^2 \iint \mathrm{d}\boldsymbol{R}_1 \iint \mathrm{d}\boldsymbol{R}_2 \left\langle a\left(\tau + \frac{2h(\boldsymbol{R}_1)}{c} - \frac{R_2^2}{cZ}\right) a\left(\tau + \frac{2h(R_2)}{cZ} - \frac{\boldsymbol{R}_2^2}{cZ}\right) \times \exp[-2\mathrm{i}k(h(\boldsymbol{R}_1) - h(\boldsymbol{R}_2))]\right\rangle \times \exp[(\mathrm{i}k/Z)(R_1^2 - R_2^2)]. \tag{21}$$

This double integral is dominated by values of $\boldsymbol{R}_1$ close to $\boldsymbol{R}_2$, because when $\boldsymbol{R}_1-\boldsymbol{R}_2$ is too large then ensemble averaging causes the exponential involving $h(\boldsymbol{R}_1)-h(\boldsymbol{R}_2)$ to vanish by destructive interference, which occurs whenever $|h(\boldsymbol{R}_1)-h(\boldsymbol{R}_2)|$ exceeds λ. But by virtue of quasi-monochromaticity the envelope factors a hardly change for such values of $\boldsymbol{R}_1-\boldsymbol{R}_2$, so that the average involving a can be evaluated separately in terms of the linked 'geographic' variations of $h(\boldsymbol{R}_1)$ and $h(\boldsymbol{R}_2)$. Defining $\boldsymbol{R}\equiv\boldsymbol{R}_1-\boldsymbol{R}_2$ and using (10) we thus obtain

$$I(\tau)=\left(\frac{k}{2\pi Z^2}\right)^2\iint \mathrm{d}\boldsymbol{R}_1\left\langle a^2\left(\tau+\frac{2h(\boldsymbol{R}_1)}{c}-\frac{R_1^2}{cZ}\right)\right\rangle$$
$$\times\iint \mathrm{d}\boldsymbol{R}\exp(-2k^2\Delta(\boldsymbol{R}))\exp\left(\frac{ik}{Z}2\boldsymbol{R}_1\cdot\boldsymbol{R}\right). \qquad (22)$$

A more detailed justification for this approximation is given in appendix 1. The next steps are to evaluate the angular integration in the $\boldsymbol{R}_1$ plane. Define a new variable

$$\tau'\equiv R_1^2/cZ \qquad (23)$$

and realise that the process of ensemble averaging over $h(\boldsymbol{R}_1)$ is independent of $\boldsymbol{R}_1$. This gives

$$I(\tau)=\pi cZ\left(\frac{k}{2\pi Z^2}\right)^2\int_0^\infty \mathrm{d}\tau'\langle a^2(\tau-\tau'+2h/c)\rangle\iint \mathrm{d}\boldsymbol{R}\exp(-2k^2\Delta(\boldsymbol{R}))J_0\{2kR\sqrt{c\tau'/Z}\} \qquad (24)$$

where J_0 denotes the zero-order Bessel function of the first kind.

Because of the Gaussian randomness of h, the remaining average is

$$\langle a^2(\tau-\tau'+2h/c)\rangle=\frac{1}{\sqrt{2\pi}H}\int_{-\infty}^{\infty}\mathrm{d}h\,a^2(\tau-\tau'+2h/c)\exp(-h^2/2H^2). \qquad (25)$$

For the pulse (14), this gives

$$\langle a^2(\tau-\tau'+2h/c)\rangle=a_0^2\frac{\sigma}{\sigma_1}\exp\left(\frac{-2c^2t^2}{\sigma_1 2}\right) \qquad (26)$$

where σ_1 is a roughness-broadened pulse length defined as

$$\sigma_1\equiv(\sigma^2+16H^2)^{1/2}. \qquad (27)$$

Despite the approximations leading to (22) and (24), these formulae give the correct flat-mirror intensity corresponding to (19) when $h=\Delta=0$. Moreover, even when the surface is not flat the formulae give for the total echo power the result

$$\int_{-\infty}^{\infty}\mathrm{d}\tau\,I(\tau)=\frac{1}{4Z^2}\int_{-\infty}^{\infty}\mathrm{d}\tau\,a^2(\tau). \qquad (28)$$

This shows that on the average a rough surface returns the same total power as a flat surface, a result expected to hold paraxially (i.e. when no power is 'sideways' lost in surface waves excited at glancing incidence, possibly after multiple scattering). Of course the distribution of this power within the echo does depend on the nature of the surface roughness, as we shall see in § 4.

For isotropic roughness, Δ depends only on R, and (24) becomes

$$I(\tau)=\frac{k^2}{2Z^3}c\int_0^\infty d\tau'\langle a^2(\tau-\tau'+2h/c)\rangle\int_0^\infty dR\,R\,\exp(-2k^2\Delta(R))J_0\{2kR\sqrt{c\tau'/Z}\} \quad (29)$$

(isotropic).

For a corrugated surface, Δ depends only on X and (22) gives

$$I(\tau)=\frac{k\sqrt{cZ}}{2\pi Z^3}\int_0^\infty \frac{d\tau'}{\sqrt{\tau'}}\langle a^2(\tau-\tau'+2h/c)\rangle\int_0^\infty dX\,\exp(-2k^2\Delta(X))\cos\{2kX\sqrt{c\tau'/Z}\}. \quad (30)$$

In evaluating these integrals, an important simplification is to replace $\Delta(\boldsymbol{R})$ by its fractal limiting form (3) for $\boldsymbol{R}\to 0$, thus achieving a 'pure diffractal' representation independent of geography (apart from the trivial H dependence of σ_1 in equation (27)). On the basis of the model (5), it is not hard to show that the decay of the exponential in (24) means that this replacement is valid if

$$Hk>1 \quad (31)$$

i.e. if the geographical fluctuations in $h(\boldsymbol{R})$ extend beyond the wavelength scale. We assume henceforth that this is the case.

4. Echo tails

The results (24), and the particular cases (29) and (30), hold whether the surface is smooth or fractal. To extract the characteristic diffractal properties it is necessary to consider time delays τ which exceed the roughness-broadened pulse duration σ_1/c (equation (27)). Then the integration variable τ' is restricted to a range σ_1/c on either side of τ. Over this range, the argument of the cosine in (30) for the 'corrugated' case varies by

$$\delta(2kX\sqrt{c\tau'/Z})=kX(2\sigma_1/c)\sqrt{c/\tau'Z}\ll 2kX\sqrt{\sigma_1/Z}. \quad (32)$$

But X is limited by the exponential decay in (30) as studied in appendix 1 (cf (A1.5)) and this leads to

$$\delta(2kX\sqrt{c\tau'}/Z)\ll 2[(kL)^{(D-1)/(2-D)}(Z/\sigma_1)^{1/2}]^{-1} \quad (33)$$

which is small in the short-wave limit and in the Fraunhofer limit. Therefore in the echo tail the cosine hardly varies with τ' over the pulse duration, and (30) can be replaced by

$$I(\tau)\overset{c\tau>\sigma_1}{\approx}\frac{k}{2\pi Z^3}\sqrt{\frac{cZ}{\tau}}\left(\int_{-\infty}^\infty dt\langle a^2(t+2h/c)\rangle\right)\int_0^\infty dX\,\exp(-2k^2\Delta(X))\cos(2kX\sqrt{c\tau/Z}). \quad (34)$$

For the 'isotropic' case, similar arguments based on the Bessel factor in (29) lead to

$$I(\tau)\overset{c\tau>\sigma_1}{\approx}\frac{k^2c}{2Z^3}\left(\int_{-\infty}^\infty dt\langle a^2(t+2h/c)\rangle\right)\int_0^\infty dR\,R\,\exp(-2k^2\Delta(R))J_0\{2kR\sqrt{c\tau/Z}\} \quad (35)$$

(isotropic).

These integrals must be evaluated for large τ. Consider first the 'corrugated' case. This depends on

$$Q(p) \equiv \int_0^\infty du \exp(-2u^{4-2D}) \cos(2up) \tag{36}$$

involving the parameter

$$p \equiv (c\tau/Z)^{1/2}/(kL)^{(D-1)/(2-D)}. \tag{37}$$

By deforming the integration contour and expanding the exponential in (36), it is possible to derive the following series for $Q(p)$:

$$Q(p) = \frac{1}{2p} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin[n\pi(2-D)][n(4-2D)]!\, 2^{n(2D-3)}}{n!\, p^{n(4-2D)}}. \tag{38}$$

This is a convergent series if $D > 1.5$, and an asymptotic one if $D < 1.5$.

For large p the series can be replaced by its first term, giving the following echo-tail formula:

$$I(\tau) \xrightarrow{\text{large } \tau} \frac{c}{2\pi Z^3}\left(\int_{-\infty}^{\infty} dt \langle a^2(t+2h/c)\rangle\right) \times \left(\frac{Z}{c\tau}\right)^{3-D} (kL)^{2D-2} \sin[\pi(2-D)](4-2D)!\, 2^{2D-4} \tag{39}$$

(corrugated).

This is our main result. It shows that in the echo tail the intensity decays as

$$I(\tau) \sim 1/\tau^{3-D}. \tag{40}$$

In speaking of the echo tail it is important to avoid a clash of asymptotics between the condition $p \gg 1$, on which the result (39) depends, and the paraxiality condition (18), according to which $c\tau$ must not exceed Z. It is clear from (37) that inconsistency is avoided by restricting attention to the *weak fractal case* where

$$kL \ll 1. \tag{41}$$

The same condition for paraxial diffractals was found by Berry (1979) for the purely monochromatic case.

To appreciate the significance of the results (39) and (40), we compare them with the echo tail from a smooth corrugated surface ($D = 1$). As $D \to 1$ the right-hand side of (39) vanishes, as do all terms in the asymptotic expansion (38). In this case, however, the integral in (34) can be evaluated exactly using the limiting form (7), to give

$$I(\tau) \xrightarrow{\text{large } \tau} \frac{c}{4Z^3\beta}\left(\int_{-\infty}^{\infty} dt \langle a^2(t+2h/c)\rangle\right)\left(\frac{Z}{2\pi c\tau}\right)^{1/2} \exp\left(-\frac{c\tau}{2Z\beta^2}\right) \tag{42}$$

(corrugated, smooth).

Therefore the intensity decays exponentially for smooth random corrugated surfaces. Notice that this result is independent of λ, illustrating the well known fact that the short-wave limit of reflection from smooth random surfaces is given by geometrical optics. By contrast, as k increases for a diffractal the echo tail (39) grows in strength until (41) is violated and the reflection is no longer paraxial.

Now we consider the 'isotropic' case. This is very similar to the 'corrugated' case. The echo tail (35) depends on

$$S(p) \equiv \int_0^\infty du\, u \exp(-2u^{4-2D}) J_0(2up)$$
$$= \frac{1}{2\pi p^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin[n\pi(2-D)]\{[n(2-D)!]^2\}2^n}{n!\, p^{n(4-2D)}} \qquad (43)$$

which is the analogue of (36)–(38). Replacing the series by its first term for large p gives the echo-tail formula

$$I(\tau) \xrightarrow{\text{large } \tau} \frac{c}{2\pi Z^3}\left(\int_{-\infty}^{\infty} dt \langle a^2(t+2h/c)\rangle\right)\left\{\frac{Z}{c\tau}\right\}^{3-D} (kL)^{2D-2} \sin[\pi(2-d)][(2-D)!]^2$$
$$\text{(isotropic)}. \qquad (44)$$

This is identical with the 'corrugated' formula (39) apart from numerical factors; in particular, the tail decays according to the power law (40).

Once again the diffractal behaves very differently from the wave diffracted by a smooth random surface, for which (35) can be evaluated exactly using the limiting form (7), to give

$$I(\tau) \xrightarrow{\text{large } \tau} \frac{c}{8\beta^2 Z^3}\left(\int_{-\infty}^{\infty} dt \langle a^2(t+2h/c)\rangle\right) \exp\left(\frac{-c\tau}{2Z\beta^2}\right) \qquad (45)$$
$$\text{(isotropic, smooth)}.$$

Again the decay is exponential, but without the additional factor $\tau^{-1/2}$ that was present for corrugated surfaces.

5. Discussion

We have found that the average power $I(\tau)$ reflected by a fractal random surface has a functional form very different from that reflected by a smooth random surface. Moreover, at least in the 'weak fractal' case where the echo is paraxial, the decay of the echo tail is a power law (equation (40)) whose exponent depends on the dimension $D+1$ of the surface and is independent of whether the surface is statistically isotropic or corrugated. The D dependence of the echo suggests that the theory could be used in conjunction with remote-sensing experiments to infer the dimensionality of naturally occurring rough surfaces such as landscapes, which as shown by Mandelbrot (1977) are well described as random fractals. Such an application would have to take account of possible effects of polarisation (not discussed in the scalar-wave theory presented here) and variable surface reflectivity (discussed by Berry (1973)).

In the past (Beckmann and Spizzichino 1963), studies of reflection from random surfaces have concentrated on the case where the roughness has a single scale (the correlation length). More recently, in studies of the monochromatic case (Gochelashvily and Shishov 1975, Rumsey 1975, Marians 1975, Furuhama 1975), it has been realised that multiscale roughness often gives better models for natural surfaces. We believe that the physical content of diffraction theory involving multiscale surfaces is much clarified by the geometric insight that comes from employing fractal terminology and emphasising the meaning of the parameter D as a dimension.

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Appendix

This is the discussion of conditions for the validity of approximation (22). Comparison with (21) shows that, in addition to the decoupling of 'slow' and 'fast' averages already discussed, this approximation consists in neglecting the following terms: in the final exponential of (21),

$$T_1 \equiv kR^2/Z \tag{A1}$$

in the second amplitude factor of (21),

$$T_2 \equiv R^2/cZ \tag{A2}$$

and

$$T_3 \equiv 2\boldsymbol{R}_1 \cdot \boldsymbol{R}/cZ; \tag{A3}$$

T_1 is dimensionless, and T_2 and T_3 are times.

R is bounded by the decay of the exponential in (22), according to which

$$k^2\Delta(\boldsymbol{R}) < 1. \tag{A4}$$

Using (3) we obtain from this the condition

$$R < L/(kL)^{1/(1-D)}. \tag{A5}$$

T_1 is now seen to satisfy

$$T_1 < \frac{L}{Z(kL)^{D/(2-D)}} \tag{A6}$$

which is negligible in the 'Fraunhofer limit' $Z \to \infty$.

The times T_2 and T_3 must be compared with the times σ/c over which the amplitude factor varies significantly. For T_2 we have

$$\frac{cT_2}{\sigma} = \frac{R^2}{Z\sigma} < \frac{L^2}{Z\sigma(kL)^{2/(2-D)}} \tag{A7}$$

which is negligible in the Fraunhofer limit, and also for long pulses ($\sigma/L \gg 1$). T_3 satisfies

$$\frac{cT_3}{\sigma} = \frac{2R_1R}{Z\sigma} < \frac{2R_1R}{Z\sigma} < \frac{2LR_1}{Z\sigma(kL)^{1/(2-D)}}. \tag{A8}$$

R_1 is limited by the paraxiality condition (18) (because $\tan\theta = R_1/Z$) and must satisfy $R_1 < Z$. Therefore cT_3/σ is negligible for long pulses. (These equations show that T_1, T_2 and T_3 are also negligible in the short-wave limit of large kL, but this conflicts with the paraxiality condition (41).)

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